Journal of Statistical Physics, Vol. 124, No. 5, September 2006 (© 2006)
DOI: 10.1007/s10955-006-9049-3

Entropic Repulsion for the High Dimensional Gaussian Lattice Field Between Two Walls

Hironobu Sakagawa[1]

Received May 18, 2005; accepted January 25, 2006
Published Online: July 27, 2006

The main aim of this paper is to discuss the entropic repulsion of random interfaces between two hard walls. We consider the $d\,(\geq 3)$-dimensional Gaussian lattice field on $\mathbb{R}^{\Lambda_N}$, $\Lambda_N = [-N, N]^d \cap \mathbb{Z}^d$ and identify the repulsion of the field as $N \to \infty$ under the condition that the field lies between two hard walls at the height level 0 and L in Λ_N where L is large enough but finite. We also study the same problem for two layered interfaces case.

KEY WORDS: entropic repulsion, Gaussian field, hard wall, random interface, Gibbs measure.

1. INTRODUCTION

1.1. Model and Result

Let $d \geq 3$, $\Lambda_N = [-N, N]^d \cap \mathbb{Z}^d$. For a configuration $\phi = \{\phi_x\}_{x \in \Lambda_N} \in \mathbb{R}^{\Lambda_N}$, we consider the following massless Hamiltonian with quadratic interaction potential:

$$H_N^{\psi}(\phi) = \frac{1}{8d} \sum_{\substack{\{x,y\} \cap \Lambda_N \neq \phi \\ |x-y|=1}} (\phi_x - \phi_y)^2 \Big|_{\phi_x \equiv \psi_x \text{ for every } x \in \partial^+ \Lambda_N},$$

where $\partial^+ \Lambda_N = \{x \notin \Lambda_N; |x-y| = 1 \text{for some } y \in \Lambda_N\}$ and $\psi = \{\psi_x\}_{x \in \partial^+ \Lambda_N}$ denotes the boundary conditions. The corresponding Gibbs measure is defined by

$$P_N^{\psi}(d\phi) = \frac{1}{Z_N^{\psi}} \exp\{-H_N^{\psi}(\phi)\} \prod_{x \in \Lambda_N} d\phi_x. \tag{1.1}$$

[1] Department of Mathematics, Faculty of Science and Technology, Keio University, 3-14-1 Hiyoshi, Kouhoku-ku, Yokohama 223-8522, Japan; e-mail: sakagawa@math.keio.ac.jp

$d\phi_x$ denotes Lebesgue measure on $\mathbb{R}$ and Z_N^ψ is a normalization factor. By summation by parts, this model coincides with a Gaussian lattice field on $\mathbb{R}^{\Lambda_N}$ whose covariance matrix is given by the inverse of a discrete Laplacian on Λ_N with Dirichlet boundary conditions outside Λ_N and this model is called harmonic crystal or lattice free field. The configuration ϕ is interpreted as an effective modelization of a random phase separating interface embedded in $d+1$-dimensional space. The spin ϕ_x at site $x \in \Lambda_N$ denotes the height of the interface.

One of the problems related to such interface is the study of the effect of the presence of a hard wall. The phenomenon arising is called entropic repulsion and is a problem to study how high an interface is pushed up by a hard wall. Such repulsion is caused by the random fluctuation of the interface naturally arises from the Lebesgue measure $d\phi$ in the Gibbs measure (1.1), in other words, by entropic effects of the measure. For the case that a hard wall is settled at the height level 0, this problem has been studied by a number of authors (cf. refs. 10 and 15 and references theirin). In $d \geq 3$, it has been proved that the interface is repelled to the level $\sqrt{4G \log N}$ as $N \to \infty$ where $G = (-\Delta)^{-1}(0, 0)$ and Δ is a discrete Laplacian on $\mathbb{Z}^d$ (cf. ref. 4). Also, the same problem for multi-layered interfaces case has been studied recently. In the case of two layered interfaces above a hard wall, it has been proved that the lower interface is repelled by a wall at the level 0 to the same height as when the upper interface is absent, in other words there is no push down effect by the upper interface to the lower one and the repulsion between two layered interfaces becomes larger than the case of an interface above a hard wall. Especially, when the covariance of two Gaussian fields are the same, the ratio of two repulsions between the hard wall and the lower interface, the lower and the upper interface is $1 : \sqrt{2}$ (cf. refs. 1 and 13).

The main aim of the present paper is to discuss what happens if we settle another hard wall above the interface, namely we are interested in the behavior of the interface which lies between two hard walls. This problem was first investigated in ref. 6 and also ref. 14 gave some probability estimates. However the result about the pathwise behavior of the interface as one wall cases has not been obtained. In this problem, different from the one wall case, both the upper and the lower walls prevent the fluctuation of a interface. Therefore the different behavior of the interface would be expected.

Now, we are in the position to state the result of this paper. We first consider the case where one interface lies between two hard walls. The corresponding event is given by

$$\mathcal{W}_N(0, L) = \{0 \leq \phi_x \leq L \quad \text{for every} \quad x \in \Lambda_N\}.$$

Throughout this paper, we always assume that the boundary conditions $\psi = \{\psi_x\}_{x \in \partial^+ \Lambda_N}$ also satisfy this two walls condition, namely we assume that $0 \leq$

$\psi_x \leq L$ for every $x \in \partial^+ \Lambda_N$. Our first result is on the probability estimate of this event.

Proposition 1.1. *For every $\gamma > 0$, there exists $L' > 0$ large enough such that the following holds: for every $L \geq L'$, there exists $N' = N'(L)$ and it holds that*

$$e^{-N^d e^{-(\frac{1}{8G}-\gamma)L^2}} \leq P_N^{\psi}(\mathcal{W}_N(0, L)) \leq e^{-N^d e^{-(\frac{1}{8G}+\gamma)L^2}}, \tag{1.2}$$

for every $N \geq N'$.

This estimate identifies the order of the exponential decay of the probability (see also Remark 1.2 below) and is sufficient to prove the repulsion phenomenon. That is, by using this probability estimate we can obtain the following asymptotics of the field under the conditional measure $P_N^{\psi}(\,\cdot\,|\mathcal{W}_N(0, L))$ under the limit $N \to \infty$. The result implies that even though the upper wall stays finite, for large enough L, the repulsion by both the upper and the lower walls keeps the interface at the height $\frac{1}{2}L$, exactly the middle level between two walls and this does not depend on the boundary conditions. For every $A \subset \mathbb{Z}^d$, $|A|$ denotes its cardinality.

Theorem 1.1. *For every $\delta > 0$ and $\eta > 0$, there exists $L' > 0$ large enough such that for every $L \geq L'$ the following holds:*

$$\lim_{N\to\infty} P_N^{\psi}\left(\left|\left\{x \in \Lambda_N; \left| \frac{\phi_x}{L} - \frac{1}{2} \right| \geq \delta\right\}\right| \geq \eta|\Lambda_N| \;\middle|\; \mathcal{W}_N(0, L)\right) = 0. \tag{1.3}$$

Next we consider the repulsion of two layered interfaces between two walls. Let $\phi^i = \{\phi_x^i\}_{x\in\Lambda_N}$, $i = 1, 2$ be two independent Gaussian fields on $\mathbb{R}^{\Lambda_N}$ distributed by (1.1) with boundary conditions $\psi^i = \{\psi_x^i\}_{x\in\partial^+\Lambda_N}$, $i = 1, 2$, respectively. Consider the event

$$\mathcal{W}_N^2(0, L) = \{0 \leq \phi_x^1 \leq \phi_x^2 \leq L \quad \text{for every} \quad x \in \Lambda_N\}.$$

In this case, we have the following:

Theorem 1.2. *For every $\delta > 0$ and $\eta > 0$, there exists $L' > 0$ large enough such that for every $L \geq L'$ the following holds:*

$$\lim_{N\to\infty} P_N^{\psi^1} \otimes P_N^{\psi^2}\left(\left|\left\{x\in\Lambda_N; \left|\frac{\phi_x^1}{L} - \left(1 - \frac{\sqrt{2}}{2}\right)\right| \geq \delta\right\}\right| \geq \eta|\Lambda_N| \;\middle|\; \mathcal{W}_N^2(0, L)\right) = 0, \tag{1.4}$$

$$\lim_{N\to\infty} P_N^{\psi^1} \otimes P_N^{\psi^2} \left(\left| \left\{ x \in \Lambda_N; \left| \frac{\phi_x^2}{L} - \frac{\sqrt{2}}{2} \right| \geq \delta \right\} \right| \geq \eta |\Lambda_N| \mathcal{W}_N^2(0, L) \right) = 0. \tag{1.5}$$

This result yields that the ratio of three repulsions between the lower wall at 0 and the lower interface ϕ^1, two interfaces ϕ^1 and ϕ^2, the upper interface ϕ^2 and the upper wall at L is $1 : \sqrt{2} : 1$. Therefore we can see that even though we settle the upper wall at finite level above two interfaces, the ratio of the repulsion between two interfaces and between a wall and an interface is preserved as the case that the upper wall is absent.

1.2. Strategy of the Proof and Several Remarks

Roughly speaking, for the one wall case, entropic repulsion is a problem to examine the enough height that an interface can fluctuate almost freely without feeling the constraint by a hard wall. It is known that in $d \geq 3$ there is a localization/delocalization transition of the interface. That is, originally the interface has a finite variance uniformly in N and by the hard wall at level 0 it is repelled to the level $\sqrt{4G \log N}$ as $N \to \infty$ everywhere above the wall. Moreover, apart from this translation effect, the shape of the interface does not change too much (cf. refs. 4 and 8).

On the contrary, if we put another hard wall above the interface then the interface becomes always localized and cannot fluctuate freely. We need to give a proper realization of such situation for the lower bound estimate of the probability of this two walls event. For this purpose, we consider a reduction of the fluctuation of the field by pinning to level 0. As the number of the pinning increases, the fluctuation of the interface becomes smaller, hence the size of the repulsion from the hard wall becomes smaller. If we can reduce the size of the repulsion by the lower wall from $\sqrt{4G \log N}$ to $\frac{1}{2}L$, then the upper wall at the level L does not play a role of a wall to the interface so much and the event that the interface lies less than level L occurs with to some extent large probability under the positively conditioned pinned measure. Then, we can adapt the previous technique for the one wall case problem to the pinned measure. Certainly we are not allowed to pin the field freely. Such pinning needs energetic and entropic cost. However, we can show that if we pin the field separately in an appropriate mesoscopic scale, then the pinning does not need too much cost and is sufficiently effective to suppress the fluctuation. As a result, we can proceed this strategy.

The upper bound estimate of the probability is also proved by considering pinning which is based on the idea of refs. 3 and 14. We first symmetrize the event $\mathcal{W}_N(0, L)$ to $\mathcal{W}_N(-\frac{1}{2}L, \frac{1}{2}L)$ by shifting the boundary conditions. This symmetry allows us pinning of the field to level 0 and we can divide Λ_N into disjoint mesoscopic scale boxes by 0 boundary conditions. Then, by Markov property of the field and the result of ref. 4 for each mesoscopic scale box, taking an appropriate scale yields the result.

As regards the height estimate, Proposition 0 gives a precise estimate of the denominator of the conditional probability $P_N^{\psi}(\,\cdot\,|\mathcal{W}_N(0, L))$ and the repulsion phenomena Theorem 1.1 can be proved by the conditioning argument for the one wall case by considering the repulsion from below and above separately.

Finally, our pinning argument also works well for the two layered interfaces case. In this case we control the repulsion of the lower interface by the lower wall and the upper interface by the upper wall, separately. We consider pinning of the lower interface to level 0 as before and for the upper interface, we first shift its mean to L and then consider pinning to level L. Since the repulsion between two interfaces can be regarded as the repulsion of one interface above a hard wall by taking its difference, we can prove the repulsion phenomena in the similar manner to the one interface case.

Next, we give several remarks about the result.

Remark 1.1. In the case of one wall settled at the level 0, pointwise estimate of the repulsion is obtained by iterating FKG argument from the density estimate of the repulsion such as Theorem 1.1 (cf. Sec. 4 of ref. 4 and Sec. 3 of ref. 7). On the other hand in the two walls case, since events $\{\phi; \phi_x \geq 0 \text{ for every } x \in \Lambda_N\}$ and $\{\phi; \phi_x \leq L \text{ for every } x \in \Lambda_N\}$ are negatively correlated, we cannot proceed such FKG argument and hence cannot obtain the pointwise estimate. Characterization of the repulsion in terms of the sample mean of the field can be easily obtained by Theorem 1.1

Remark 1.2. With respect to the probability estimate, ref. 6 showed the similar asymptotics to Proposition 1.1 without identifying the exact constant $\frac{1}{8G}$ in (1.2) and ref. 14 studied this problem in the context of non Gaussian massless fields with strictly convex interactions.

In the case of $d = 2$, refs. 6 and 14 showed that the order L^2 in (1.2) should be replaced by L. However the exact constant in the asymptotics of the probability and the path behavior under the conditoned measure are still unknown.

Remark 1.3. We took the height L of the upper interface independent of the size of the system N. In the one interface problem with 0 boundary conditions, we can consider the case that $L = L_N$ depends on N, namely the position of the upper wall rises up at the same time the size N of the system grows. In this case, if $\limsup_{N\to\infty} \frac{L_N}{\sqrt{\log N}} \leq 2\sqrt{4G}$ then we obtain the same result as Theorem 1.1 and the interface is repelled to the level $\frac{1}{2}L_N$. Also if $\liminf_{N\to\infty} \frac{L_N}{\sqrt{\log N}} > 2\sqrt{4G}$ then the interface is repelled to the level $\sqrt{4G \log N}$. That is, the upper wall rises up too fast and it does not play a role of a wall.

The rest of the paper is divided into 3 sections. Sec. 2 gives the proof of probability estimates Proposition 1.1. In Sec. 3, we prove the repulsion phenomena Theorem 1.1. Finally, two layered interfaces case is studied in Sec. 4. We remark that throughout this paper below, C represents a positive constant which does not depend on L, N but may depend on other parameters. Also, this C in estimates may change from place to place in the paper.

2. PROBABILITY ESTIMATES

In this section, We shall prove Proposition 1.1 The following lemma allows us considering only the case with flat boundary conditions.

Lemma 2.1. *Assume that $0 \le \psi_x \le L$ for every $x \in \partial^+\Lambda_N$. Then, there exists some constant $C > 0$ independent of L, N such that*

$$\begin{aligned} &e^{-CN^{d-1}L^2} P_N^h(\mathcal{E} \cap \mathcal{W}_N(0, L)) \\ &\quad \le P_N^{\psi}(\mathcal{E} \cap \mathcal{W}_N(0, L)) \le e^{CN^{d-1}L^2} P_N^h(\mathcal{E} \cap \mathcal{W}_N(0, L)), \end{aligned}$$

for every $0 \le h \le L$ and event $\mathcal{E}$, where P_N^h denotes the Gibbs measure (1.1) with flat h boundary conditions.

Proof: By dividing the summation of the Hamiltonian into the interior part and the boundary part of Λ_N, we have

$$H_N^{\psi}(\phi) - H_N^h(\phi) = \frac{1}{4d} \sum_{\substack{x\in\Lambda_N, y\in\partial^+\Lambda_N \\ |x-y|=1}} \{(\phi_x - \psi_y)^2 - (\phi_x - h)^2\}.$$

Therefore, under the conditions $0 \le \psi_x \le L$ for every $x \in \partial^+\Lambda_N$, $\mathcal{W}_N(0, L)$ and $0 \le h \le L$, we have

$$-CN^{d-1}L^2 \le H_N^{\psi}(\phi) - H_N^h(\phi) \le CN^{d-1}L^2, \tag{2.1}$$

for some constant $C > 0$. This estimate also yields that

$$e^{-CN^{d-1}L^2} Z_N^h \le Z_N^{\psi} \le e^{CN^{d-1}L^2} Z_N^h. \tag{2.2}$$

Note that Z_N^h is independent of h. By (2.1) and (2.2) we complete the proof. □

For the proof of the lower bound, we take 0 boundary conditions. In this case, we omit to write the boundary conditions and simply denote as P_N, Z_N etc. P_N coincides with the law of the centered Gaussian field on $\mathbb{R}^{\Lambda_N}$ with covariance matrix $(-\Delta_N)^{-1}$. Let $\theta = \theta(L)$ be a mesoscopic scale which goes to infinity as $L \to \infty$ (we will choose θ in the sequel) and define $\Gamma_A = [\theta]\mathbb{Z}^d \cap A$ for $A \subset \mathbb{Z}^d$. Γ_{Λ_N} is denoted as Γ_N. We consider the pinning of each site of Γ_N to level 0. The corresponding Gibbs measure is given by

$$Q_N(\,\cdot\,) \equiv P_{\Lambda_N \setminus \Gamma_N}(\,\cdot\,) = P_N(\,\cdot\,|\phi_x \equiv 0 \quad \text{for every} \quad x \in \Gamma_N).$$

This is a law of the centered Gaussian field on $\mathbb{R}^{\Lambda_N \setminus \Gamma_N}$ with covariance matrix $(-\Delta_{\Lambda_N \setminus \Gamma_N})^{-1}$ and 0 boundary conditions outside $\Lambda_N \setminus \Gamma_N$. We first estimates the cost of the pinning.

Lemma 2.2. *There exists a constant $C > 0$ such that*

$$P_N\big(\mathcal{W}_N(0, L)\big) \geq e^{-CN^d\theta^{-d}L^2} Q_N\big(\mathcal{W}_N(0, L)\big),$$

for every L, N large enough.

Proof: By decomposing summation in $H_N(\phi)$ into nearest neighbor pairs which include a point of Γ_N or not, we can easily calculate that

$$H_N(\phi) = H_{\Lambda_N \setminus \Gamma_N}(\phi) + \frac{1}{4d} \sum_{x \in \Gamma_N} \sum_{\substack{y \in \Lambda_N \\ |y-x|=1}} \big(\phi_x^2 - 2\phi_x\phi_y\big).$$

The second term in the right hand side is less than $CN^d\theta^{-d}L^2$ for some constant $C > 0$ under the condition $\mathcal{W}_N(0, L)$. Therefore, we have

$$\begin{aligned} &P_N(\mathcal{W}_N(0, L)) \\ &\geq \frac{1}{Z_N} \int_{\mathbb{R}^{\Lambda_N}} I\big(\mathcal{W}_N(0, L)\big) e^{-H_{\Lambda_N \setminus \Gamma_N}(\phi)} e^{-CN^d\theta^{-d}L^2} \prod_{x \in \Lambda_N \setminus \Gamma_N} d\phi_x \prod_{x \in \Gamma_N} d\phi_x \\ &= e^{-CN^d\theta^{-d}L^2} L^{|\Gamma_N|} \frac{Z_{\Lambda_N \setminus \Gamma_N}}{Z_N} Q_N(\mathcal{W}_N(0, L)). \end{aligned}$$

Now, by Lemma 2.3.1 (a) of ref. 5 (note that the argument given there can be extended to all $d \geq 1$), we know that $Z_{\Lambda_N \setminus \Gamma_N} \geq Ce^{-C|\Gamma_N|} Z_N$ for some constant $C > 0$ and this completes the proof. □

By this lemma, our problem is reduced to the lower bound estimate of probability of two walls event under the pinned measure Q_N and this can be decomposed as follows:

$$Q_N\big(\mathcal{W}_N(0, L)\big) = Q_N\big(\Omega_N^+(0)\big) Q_N\big(\Omega_N^-(L)\big|\Omega_N^+(0)\big), \tag{2.3}$$

where

$$\Omega_N^+(0) = \{\phi_x \geq 0 \quad \text{for every} \quad x \in \Lambda_N\},$$
$$\Omega_N^-(L) = \{\phi_x \leq L \quad \text{for every} \quad x \in \Lambda_N\}.$$

We estimate each term in the right hand side from below. The first term is a problem of the entropic repulsion for the pinned measure Q_N.

Lemma 2.3. *There exists a constant $C > 0$ such that the following holds: for every $L > 0$ large enough, there exists $N' = N'(L)$ and it holds that*

$$Q_N\big(\Omega_N^+(0)\big) \geq e^{-CN^d\theta^{-d}\log\theta},$$

for every $N \geq N'$.

Proof: We use the well-known measure change argument (cf. Sec. 3.5 of ref. 10). For $\lambda > 0$, let Q_N^λ be the law of the Gaussian field on $\mathbb{R}^{\Lambda_N}$ with mean λ and covariance matrix $(-\Delta_{\Lambda_N\setminus\Gamma_N})^{-1}$. We also define $Q_N^{\lambda,+}(\,\cdot\,) \equiv Q_N^\lambda(\,\cdot\,|\Omega_N^+(0))$. Then, we have

$$H\big(Q_N^{\lambda,+} \mid Q_N\big) = E^{Q_N^{\lambda,+}}\left[\log\frac{dQ_N^{\lambda,+}}{dQ_N^\lambda}\right] + E^{Q_N^{\lambda,+}}\left[\log\frac{dQ_N^\lambda}{dQ_N}\right] \equiv J_1 + J_2,$$

where, for two probability measures μ and ν with $\mu \ll \nu$, $H(\mu|\nu) = E^\mu\big[\log\frac{d\mu}{d\nu}\big]$ denotes the relative entropy of μ with respect to ν.

We shall estimate J_1 and J_2, respectively. At first, we have $J_1 = -\log Q_N^\lambda(\Omega_N^+(0))$. FKG inequality and Gaussian tail estimate yields $J_1 \leq CN^d e^{-\frac{\lambda^2}{2G}}$ for every λ, N large enough. For J_2, we have

$$\begin{aligned} J_2 &= E^{Q_N^{\lambda,+}}\Bigg[-\frac{1}{8d}\sum_{\{x,y\}\cap(\Lambda_N\setminus\Gamma_N)\neq\phi}(\phi_x-\phi_y)^2\,\Big|_{\phi_x\equiv\lambda \text{ for every } x\in\partial^+\Lambda_N\cup\Gamma_N} \\ &\quad + \frac{1}{8d}\sum_{\{x,y\}\cap(\Lambda_N\setminus\Gamma_N)\neq\phi}(\phi_x-\phi_y)^2\,\Big|_{\phi_x\equiv 0 \text{ for every } x\in\partial^+\Lambda_N\cup\Gamma_N}\Bigg] \\ &\leq C\lambda^2|\partial^+\Lambda_N\cup\Gamma_N| \\ &\leq C\lambda^2(N^{d-1}+N^d\theta^{-d}). \end{aligned}$$

Now taking λ as $\lambda = \sqrt{2dG\log\theta}$, we obtain

$$H\big(Q_N^{\lambda,+} \mid Q_N\big) \leq C(N^d\theta^{-d} + N^{d-1})\log\theta,$$

and finally by an entropy inequality (cf. (B. 23) of ref. 10):

$$\log \frac{\mu(A)}{\nu(A)} \geq -\frac{1}{\nu(A)}\big(H(\nu|\mu) + e^{-1}\big),$$

we can complete the proof. □

The following lemma implies that by mesoscopic scale pinning we can suppress the level of the repulsion by a wall at the level 0. The proof will be given later.

Lemma 2.4. *For every $\gamma > 0$, there exists $\theta' > 0$ large enough such that the following holds: for every $\theta \geq \theta'$, there exists $N' = N'(\theta)$ and it holds that*

$$\sup_{x\in\Lambda_N} E^{Q_N}[\phi_x|\Omega_N^+(0)] \leq \sqrt{2dG\log\theta}(1+\gamma),$$

for every $N \geq N'$.

Once we obtain this moment bound, we can easily estimate the second term of (2.3).

Lemma 2.5. *Set $\theta = e^{\frac{L^2}{8dG}}$. Then, for every $\gamma > 0$, there exists $L' > 0$ large enough such that the following holds: for every $L \geq L'$, there exists $N' = N'(L)$ and it holds that*

$$Q_N\big(\Omega_N^-(L)\big|\Omega_N^+(0)\big) \geq e^{-N^d e^{-(\frac{1}{8G}-\gamma)L^2}},$$

for every $N \geq N'$.

Proof: By using FKG inequality and Brascamp Lieb inequality (cf. Appendix B.1 and B.2 of ref.10), we see that

$$\begin{aligned} &Q_N\big(\Omega_N^-(L) \;\big|\; \Omega_N^+(0)\big) \\ &\geq \prod_{x\in\Lambda_N}\Big[1 - \exp\Big\{-\frac{\big((L - E^{Q_N}[\phi_x|\Omega_N^+(0)]) \vee 0\big)^2}{2G}\Big\}\Big]. \end{aligned}$$

Therefore, Lemma 2 with taking $\theta = e^{\frac{L^2}{8dG}}$ yields the result. □

Proof of the lower bound of Proposition 1.1. By Lemma 2.1, 2.2, 2.3, 2.5 and (2.3), taking θ as $\theta = e^{\frac{L^2}{8dG}}$ yields the lower bound of (1.2). □

The rest is to prove Lemma 2.4. For this purpose, we prepare a lemma which controls the fluctuation of the field under the pinned measure. We set $\Gamma_N = [\theta]\mathbb{Z}^d \cap \Lambda_N$ and $A + x = \{y + x; y \in A\}$ for every $A \subset \mathbb{Z}^d$, $x \in \mathbb{Z}^d$.

Lemma 2.6. *Let $0 < r_1, r_2, r_3 < 1$ with $r_1 + r_2 < r_3$. Then, there exists a constant $C > 0$ such that the following holds: for every $\theta > 0$ there exists $N' = N'(\theta)$ and it holds that*

$$\sup_{x\in\Lambda_{r_2N}} \mathrm{Var}_{P_N}\left(\sum_{z\in\Gamma_{r_1N}+x} \phi_z \,\middle|\, \phi_y \equiv 0 \quad \text{for every} \quad y \in \Gamma_{r_3N}\right) \le CN^d\theta^{-d},$$

for every $N \ge N'$.

Proof: Let $\{S_n\}_{n\ge0}$ be a simple random walk on $\mathbb{Z}^d$. $\mathbb{P}_x$ denotes its law starting at $x \in \mathbb{Z}^d$ and $\mathbb{E}_x$ denotes the corresponding expectation. Also for $A \subset \mathbb{Z}^d$, $\tau_A = \inf\{n \ge 0; S_n \in A\}$ denotes the first hitting time of A. Then, by the random walk representation, we know that

$$\begin{aligned}
&\mathrm{Var}_{P_N}\left(\sum_{z\in\Gamma_{r_1N}+x} \phi_z \,\middle|\, \phi_y \equiv 0 \quad \text{for every} \quad y \in \Gamma_{r_3N}\right)\\
&\quad = \sum_{y,z\in\Gamma_{r_1N}+x} \mathbb{E}_y\left[\sum_{n=0}^{\infty} I(S_n = z, n < \tau_{\partial^+\Lambda_N} \wedge \tau_{\Gamma_{r_3N}})\right]\\
&\quad = \sum_{y\in\Gamma_{r_1N}+x} \mathbb{E}_y\left[\sum_{n=0}^{\infty} I(S_n \in \Gamma_{r_1N} + x, n < \tau_{\partial^+\Lambda_N} \wedge \tau_{\Gamma_{r_3N}})\right].
\end{aligned}$$

Therefore it suffices to prove that

$$\sup_{x\in\Lambda_{r_2N}} \sup_{y\in\Gamma_{r_1N}+x} \mathbb{E}_y\left[\sum_{n=0}^{\infty} I(S_n \in \Gamma_{r_1N} + x, n < \tau_{\partial^+\Lambda_N} \wedge \tau_{\Gamma_{r_3N}})\right] \le C, \tag{2.4}$$

uniformly in N. At first, we have

$$\begin{aligned}
K_N &\equiv \sup_{y\in\Gamma_{r_1N}+x} \mathbb{E}_y\left[\sum_{n=0}^{\infty} I(S_n \in \Gamma_{r_1N} + x, n < \tau_{\partial^+\Lambda_N} \wedge \tau_{\Gamma_{r_3N}})\right]\\
&= \sup_{y\in\Gamma_{r_1N}+x}\left\{\mathbb{E}_y\left[\sum_{n=0}^{\infty} I(S_n = y, n < \tau_{\partial^+\Lambda_N} \wedge \tau_{\Gamma_{r_3N}})\right]\right.\\
&\qquad \left. + \mathbb{E}_y\left[\sum_{n=0}^{\infty} I(S_n \in (\Gamma_{r_1N} + x) \setminus \{y\}, n < \tau_{\partial^+\Lambda_N} \wedge \tau_{\Gamma_{r_3N}})\right]\right\}
\end{aligned}$$

$$\leq G + \sup_{y \in \Gamma_{r_1 N}+x} \mathbb{E}_y \left[\sum_{n=0}^{\infty} I(S_n \in (\Gamma_{r_1 N} + x) \setminus \{y\}, n < \tau_{\partial^+ \Lambda_N} \wedge \tau_{\Gamma_{r_3 N}}) \right].$$

We estimate the second term as

$$\mathbb{E}_y \left[\sum_{n=0}^{\infty} I(S_n \in (\Gamma_{r_1 N} + x) \setminus \{y\}, n < \tau_{\partial^+ \Lambda_N} \wedge \tau_{\Gamma_{r_3 N}}) \right]$$

$$= \mathbb{E}_y \left[\sum_{n=0}^{\infty} I(S_n \in (\Gamma_{r_1 N} + x) \setminus \{y\}, \right.$$

$$\left. n < \tau_{\partial^+ \Lambda_N} \wedge \tau_{\Gamma_{r_3 N}}, \tau_{(\Gamma_{r_1 N}+x)\setminus\{y\}} < \tau_{\partial^+ \Lambda_N} \wedge \tau_{\Gamma_{r_3 N}}) \right]$$

$$\leq \mathbb{P}_y(\tau_{(\Gamma_{r_1 N}+x)\setminus\{y\}} < \tau_{\partial^+ \Lambda_N} \wedge \tau_{\Gamma_{r_3 N}})$$

$$\times \sup_{z \in (\Gamma_{r_1 N}+x)\setminus\{y\}} \mathbb{E}_z \left[\sum_{n=0}^{\infty} I(S_n \in (\Gamma_{r_1 N} + x) \setminus \{y\}, n < \tau_{\partial^+ \Lambda_N} \wedge \tau_{\Gamma_{r_3 N}}) \right]$$

$$\leq \mathbb{P}_y(\tau_{(\Gamma_{r_1 N}+x)\setminus\{y\}} < \tau_{\partial^+ \Lambda_N} \wedge \tau_{\Gamma_{r_3 N}}) K_N,$$

where we used strong Markov property for the first inequality. Hence for (2.4), all we need to show is

$$\sup_{x \in \Lambda_{r_2 N}} \sup_{y \in \Gamma_{r_1 N}+x} \mathbb{P}_y(\tau_{(\Gamma_{r_1 N}+x)\setminus\{y\}} < \tau_{\partial^+ \Lambda_N} \wedge \tau_{\Gamma_{r_3 N}}) < 1,$$

uniformly in N. We decompose

$$\begin{aligned} &\mathbb{P}_y(\tau_{(\Gamma_{r_1 N}+x)\setminus\{y\}} < \tau_{\partial^+ \Lambda_N} \wedge \tau_{\Gamma_{r_3 N}}) \\ &\quad \leq \mathbb{P}_y(\tau_{(\Gamma_{r_1 N}+x)\setminus\{y\}} < \tau_{\partial^+ \Lambda_{r_3 N}} \wedge \tau_{\Gamma_{r_3 N}}) + \mathbb{P}_y(\tau_{\partial^+ \Lambda_{r_3 N}} < \tau_{\Gamma_{r_3 N}}), \end{aligned}$$

and it is obvious that the first term is less than some constant $C < 1$ uniformly in x, y and N since we got rid of the starting point of the random walk and consider killing at $\partial^+ \Lambda_{r_3 N}$. For the second term we follow the argument of Lemma A.7 of ref. 2. For every $T > 0$, we have

$$\mathbb{P}_y(\tau_{\partial^+ \Lambda_{r_3 N}} < \tau_{\Gamma_{r_3 N}}) \leq \mathbb{P}_y(T < \tau_{\partial^+ \Lambda_{r_3 N}} \wedge \tau_{\Gamma_{r_3 N}}) + \mathbb{P}_y(\tau_{\partial^+ \Lambda_{r_3 N}} \leq T).$$

By the proof of (A.8) of ref. 2, we know that

$$\mathbb{P}_y(T < \tau_{\partial^+ \Lambda_{r_3 N}} \wedge \tau_{\Gamma_{r_3 N}}) \leq \exp\{-C\theta^{-d} T\}. \tag{2.5}$$

Also by Lemma 1.5.1 of ref. 12,

$$\mathbb{P}_y(\tau_{\partial^+ \Lambda_{r_3 N}} \leq T) \leq \mathbb{P}_0(\sup_{0 \leq n \leq T} |S_n| \geq \gamma N) \leq \exp\{-C \frac{N}{\sqrt{T}}\}, \tag{2.6}$$

where $\gamma = r_3 - (r_1 + r_2) > 0$. Note that $\mathrm{dist}(\Gamma_{r_1 N} + x, \Lambda^c_{r_3 N}) \geq \gamma N$ for every $x \in \Lambda_{r_2 N}$. Now, taking T as $T = N^{\frac{2}{3}}\theta^{\frac{2d}{3}}$ in (2.5) and (2.6), we obtain

$$\mathbb{P}_y(\tau_{\partial^+ \Lambda_{r_3 N}} < \tau_{\Gamma_{r_3 N}}) \leq \exp\{-CN^{\frac{2}{3}}\theta^{-\frac{d}{3}}\},$$

and this is exponentially small for N large enough. Therefore, we complete the proof. □

Proof of Lemma 2.4. Let $0 < \varepsilon < 1$. At first, we have

$$\bigcup_{y \in \Gamma_{\frac{1}{4}\varepsilon N}} (\Lambda_N + y) \subset \Lambda_{(1+\frac{1}{4}\varepsilon)N}, \quad \bigcap_{y \in \Gamma_{\frac{1}{4}\varepsilon N}} (\Gamma_N + y) \supset \Gamma_{N, \frac{1}{4}\varepsilon},$$

where we denote $\Gamma_{N,\varepsilon} = [\theta]\mathbb{Z}^d \cap \Lambda_{N,\varepsilon}$, $\Lambda_{N,\varepsilon} = \{x \in \Lambda_N; \mathrm{dist}(x, \Lambda^c_N) \geq \varepsilon N\}$. Hence by FKG inequality (cf. (B. 10) of ref. 10), we can obtain that

$$\begin{aligned} E^{Q_N}[\phi_x | \Omega^+_N(0)] &= E^{P_{(\Lambda_N + y)\setminus(\Gamma_N + y)}}[\phi_{x+y} | \Omega^+_{\Lambda_N + y}(0)] \\ &\leq E^{P_{\Lambda_{(1+\frac{1}{4}\varepsilon)N} \setminus \Gamma_{N,\frac{1}{4}\varepsilon}}}[\phi_{x+y} | \Omega^+_{\Lambda_{(1+\frac{1}{4}\varepsilon)N}}(0)], \end{aligned}$$

for every $x \in \Lambda_N$ and $y \in \Gamma_{\frac{1}{4}\varepsilon N}$. This yields

$$E^{Q_N}[\phi_x | \Omega^+_N(0)] \leq E^{P_{\Lambda_{(1+\frac{1}{4}\varepsilon)N} \setminus \Gamma_{N,\frac{1}{4}\varepsilon}}}\Big[\frac{1}{|\Gamma_{\frac{1}{4}\varepsilon N}|} \sum_{z \in \Gamma_{\frac{1}{4}\varepsilon N} + x} \phi_z | \Omega^+_{\Lambda_{(1+\frac{1}{4}\varepsilon)N}}(0)\Big].$$

Next, set $T_N(x) \equiv \frac{1}{|\Gamma_{\frac{1}{4}\varepsilon N}|} \sum_{z \in \Gamma_{\frac{1}{4}\varepsilon N} + x} \phi_z$. By shifting the boundary conditions on $\Lambda^c_{(1+\frac{1}{4}\varepsilon)N} \cup \Gamma_{N,\frac{1}{4}\varepsilon}$ from 0 to $\lambda > 0$ and using a stochastic domination (cf. (B.13) of ref. 10), we have

$$\begin{aligned} & E^{P_{\Lambda_{(1+\frac{1}{4}\varepsilon)N} \setminus \Gamma_{N,\frac{1}{4}\varepsilon}}}\big[T_N(x) | \Omega^+_{\Lambda_{(1+\frac{1}{4}\varepsilon)N}}(0)\big] \\ &\leq \lambda + E^{P_{\Lambda_{(1+\frac{1}{4}\varepsilon)N} \setminus \Gamma_{N,\frac{1}{4}\varepsilon}}}\big[T_N(x) | \Omega^+_{\Lambda_{(1+\frac{1}{4}\varepsilon)N}}(-\lambda)\big] \\ &\equiv \lambda + J_1. \end{aligned}$$

By using the inequality (cf. (B.24) of ref. 10):

$$E^{\mu}[X | \mathcal{A}] \leq \frac{1}{t} \log E^{\mu}[e^{tX}] - \frac{1}{t} \log \mu(\mathcal{A}), \quad t > 0,$$

taking $\mu = P_{\Lambda_{(1+\frac{1}{4}\varepsilon)N} \setminus \Gamma_{N,\frac{1}{4}\varepsilon}}$, $X = \frac{T_N(x)}{\lambda}$ and $\mathcal{A} = \Omega^+_{\Lambda_{(1+\frac{1}{4}\varepsilon)N}}(-\lambda)$, we see that

$$\frac{1}{\lambda} J_1 \leq \frac{1}{t} \log E^{P_{\Lambda_{(1+\frac{1}{4}\varepsilon)N} \setminus \Gamma_{N,\frac{1}{4}\varepsilon}}}\big[\exp\{\frac{t}{\lambda} T_N(x)\}\big]$$

$$
\begin{aligned}
&- \frac{1}{t} \log P_{\Lambda_{(1+\frac{1}{4}\varepsilon)N} \setminus \Gamma_{N,\frac{1}{4}\varepsilon}} \left(\Omega^{+}_{\Lambda_{(1+\frac{1}{4}\varepsilon)N}}(-\lambda) \right) \\
&\equiv J_2 + J_3.
\end{aligned}
$$

Now, take $\lambda = \sqrt{2dG \log \theta}$ and $t = N^d \theta^{-d} \sqrt{\log \theta}$. By Lemma 2.6, variance of $T_N(x)$ under the measure $P_{\Lambda_{(1+\frac{1}{4}\varepsilon)N} \setminus \Gamma_{N,\frac{1}{4}\varepsilon}}$ is less than $CN^{-d}\theta^d$ for every $x \in \Lambda_{N,\varepsilon}$. Note that $\mathrm{dist}(\Gamma_{\frac{1}{4}\varepsilon N} + x, \Lambda^c_{N,\frac{1}{4}\varepsilon}) \geq \frac{1}{2}\varepsilon N$ for $x \in \Lambda_{N,\varepsilon}$. Hence we can obtain that $J_2 \leq \frac{C}{\sqrt{\log \theta}}$ for every θ, N large enough (N depends on θ). For J_3, FKG inequality and Gaussian tail estimate yields the same estimate as J_2. Collecting all the estimates, we complete the proof for the interior case $x \in \Lambda_{N,\varepsilon}$ for every $0 < \varepsilon < 1$. The boundary case $x \in \Lambda_N \setminus \Lambda_{N,\varepsilon}$, $0 < \varepsilon < 1$ follows from the interior case result and FKG argument. $\square$

Next, we prove the upper bound of Proposition 1.1. For every $A \subset \mathbb{Z}^d$ and $a \leq b$, P_A denotes the Gibbs measure (1.1) on A with 0 boundary conditions outside A and we set $\mathcal{W}_A(a, b) = \{\phi; a \leq \phi_x \leq b \text{ for every } x \in A\}$. Especially, in the case of $A = \Lambda_N$, we denote this event as $\mathcal{W}_N(a, b)$. The following lemma allows us pinning of the field to the level 0 under the symmetric two walls condition.

Lemma 2.7. *For every finite sets $B \subset A \subset \mathbb{Z}^d$, $x_0 \in A$ and $L > 0$, the following holds:*

$$P_A\big(\mathcal{W}_B(-L, L)\big) \leq P_{A\setminus\{x_0\}}\big(\mathcal{W}_B(-L, L)\big).$$

Proof: By decomposing summation in $H_A(\phi)$ into nearest neighbor pairs which include x_0 or not, we can easily calculate that

$$H_A(\phi) = H_{A\setminus\{x_0\}}(\phi) + \frac{1}{2}\phi_{x_0}^2 - T_{x_0}(\phi)\phi_{x_0},$$

where $T_{x_0}(\phi) = \frac{1}{2d}\sum_{y\in\mathbb{Z}^d;|y-x_0|=1} \phi_y$. By this equality, we compute that

$$
\begin{aligned}
&\frac{Z_A P_A\big(\mathcal{W}_B(-L, L)\big)}{Z_{A\setminus\{x_0\}} P_{A\setminus\{x_0\}}\big(\mathcal{W}_B(-L, L)\big)} \\
&= E^P_{A\setminus\{x_0\}}\left[\int_{\mathbb{R}} \exp\left\{-\frac{1}{2}r^2 + T_{x_0}(\phi)r\right\} dr \Big| \mathcal{W}_B(-L, L)\right] \\
&= \int_{\mathbb{R}} \exp\left\{-\frac{1}{2}r^2\right\} E^P_{A\setminus\{x_0\}}\Big[\exp\big\{T_{x_0}(\phi)r\big\}\big|\mathcal{W}_B(-L, L)\Big] dr \\
&\leq \int_{\mathbb{R}} \exp\left\{-\frac{1}{2}r^2\right\} \exp\left\{\frac{1}{2}r^2 \mathrm{Var}_{P_{A\setminus\{x_0\}}}(T_{x_0}(\phi))\right\} dr,
\end{aligned}
$$

where we used Brascamp Lieb inequality and symmetry for the last inequality. The rightmost term coincides with $\frac{Z_A}{Z_{A\setminus\{x_0\}}}$. □

Proof of the upper bound of Proposition 1.1. By Lemma 2.1 and shifting the boundary conditions, we have

$$\begin{aligned} P_N^{\psi}\big(\mathcal{W}_N(0,L)\big) &\le e^{CN^{d-1}L^2} P_N^{\frac{1}{2}L}(\mathcal{W}_N(0,L)) \\ &= e^{CN^{d-1}L^2} P_N\left(\mathcal{W}_N\left(-\frac{1}{2}L, \frac{1}{2}L\right)\right), \end{aligned}$$

for some constant $C > 0$. Now, we consider a mesoscopic scale of order $\theta = \theta(L)$ and the partition of Λ_N into boxes with the side-length $2[\theta]+1$. We take this partition so that boundaries of neighboring boxes intersect and denotes the set of the whole boundary by B_N. For simplicity, we assume that $[\theta]$ divides $N+1$. The total number of the mesoscopic scale boxes is $CN^d\theta^{-d}(1+o(1))$. By using Lemma 2.7 repeatedly and Markov property of the field, we have

$$\begin{aligned} P_N\big(\mathcal{W}_N(0,L)\big) &\le P_N\left(\mathcal{W}_N\left(-\frac{1}{2}L, \frac{1}{2}L\right) \mid \phi_x \equiv 0 \quad \text{for every} \quad x \in B_N\right) \\ &\le \left(P_{\Lambda_\theta}\left(\mathcal{W}_{\Lambda_\theta}\left(-\frac{1}{2}L, \frac{1}{2}L\right)\right)\right)^{CN^d\theta^{-d}}. \end{aligned}$$

Now, for every $\gamma > 0$, choose θ as $\theta = e^{(\frac{1}{16G}+\gamma)L^2}$. Then,

$$\begin{aligned} P_{\Lambda_\theta}\left(\mathcal{W}_{\Lambda_\theta}\left(-\frac{1}{2}L, \frac{1}{2}L\right)\right) &\le P_{\Lambda_\theta}\left(\Omega^+_{\Lambda_{\theta,\eta}}\left(-\frac{1}{2}L\right)\right) \\ &\le P_{\Lambda_\theta}\left(\Omega^+_{\Lambda_{\theta,\eta}}\left(-\sqrt{4G(1-16G\gamma/1+16G\gamma)\log\theta}\right)\right) \\ &\le e^{-C\theta^{d-2}\log\theta}, \end{aligned}$$

for large enough L, where $\Omega^+_{\Lambda_{\theta,\eta}}(l) = \{\phi_x \ge l\,,\ \text{for every } x \in \Lambda_{\theta,\eta}\}$ for $l \in \mathbb{R}$ and $\Lambda_{\theta,\eta} = \{x \in \Lambda_\theta; \mathrm{dist}(x, \Lambda_\theta^c) \ge \eta\theta\}$, $0 < \eta < 1$. The last inequality follows from the result of ref. 4. Combining these inequalities, we complete the proof. □

3. ENTROPIC REPULSION

In this section, we shall prove Theorem 1.1. We adapt the conditioning argument of Sec. 3.6 and 3.7 of ref. 10. Let $K \in \mathbb{N}$ large but fixed. $B_K(x) = \{y \in \mathbb{Z}^d; \max_{1\le j\le d} |y_j - x_j| = K\}$ denotes the boundary of a box with side-length $2K+1$ and centered at $x \in \mathbb{Z}^d$. For $z \in \widetilde{\Lambda}_K \equiv [-2K, 2K)^d \cap \mathbb{Z}^d$, we define $D_K(z) = 4K\mathbb{Z}^d + z$, $\Lambda_N^K(z) = \{x \in \Lambda_N \cap D_K(z); B_K(x) \subset \Lambda_N\}$ and

$\overline{B}_K(z) = \cup_{x\in\Lambda_N^K(z)} B_K(x)$. $\Lambda_N^K(z)$ are disjoint for each $z \in \widetilde{\Lambda}_K$ and we have $\big| \Lambda_N \setminus \cup_{z\in\widetilde{\Lambda}_K} \Lambda_N^K(z) \big| = O(N^{d-1})$. We also set $\Lambda_N^K = \Lambda_N^K(0)$, $\overline{B}_K = \overline{B}_K(0)$, and $\mathcal{F}_{\overline{B}_K} = \sigma(\phi_x; x \in \overline{B}_K)$. Under the conditional measure $P_N(\,\cdot\,|\mathcal{F}_{\overline{B}_K})$, $\{\phi_x; x \in \Lambda_N^K\}$ are independent Gaussian random variables with mean $M_x(\phi) \equiv \sum_{y\in B_K(x)} q_K(x,y)\phi_y$ and variance G^K where $q_K(x,y)$ is the probability that a simple random walk on $\mathbb{Z}^d$ starting at x hits $B_K(x)$ at y first and G^K is a $(0,0)$-coordinate of the Green function of a simple random walk starting at 0 and killed when hitting $B_K(0)$.

Lemma 3.1. *For every* $0 < \varepsilon < 1, r_1 > 2 - \frac{\sqrt{G^K}}{\sqrt{G}}, r_2 < \frac{\sqrt{G^K}}{\sqrt{G}}$*, there exists* $L' \in \mathbb{N}$ *such that for every* $L \geq L'$ *the following holds:*

$$\lim_{N\to\infty} P_N^\psi\left(\left|\left\{x \in \Lambda_N^K; \phi_x \geq \frac{1}{2}r_1 L\right\}\right| \geq \eta\left|\Lambda_N^K\right| \middle| \mathcal{W}_N(0,L)\right) = 0, \tag{3.1}$$

$$\lim_{N\to\infty} P_N^\psi\left(\left|\left\{x \in \Lambda_N^K; \phi_x \leq \frac{1}{2}r_2 L\right\}\right| \geq \eta\left|\Lambda_N^K\right| \middle| \mathcal{W}_N(0,L)\right) = 0. \tag{3.2}$$

Once we have this lemma, by shifting the partition and the corresponding set of centers, we can obtain the similar results for $\Lambda_N^K(z), z \in \widetilde{\Lambda}_K$ instead of Λ_N^K. Then, noting that

$$\begin{aligned}
&\left\{\phi; \left|\left\{x \in \Lambda_N; \left|\frac{\phi_x}{L} - \frac{1}{2}\right| \geq \delta\right\}\right| \geq \eta|\Lambda_N|\right\} \\
&\subset \bigcup_{z\in\widetilde{\Lambda}_K} \left(\left\{\phi; \left|\left\{x \in \Lambda_N^K(z); \phi_x \geq \left(\frac{1}{2}+\delta\right)L\right\}\right| \geq \frac{1}{2}\eta\left|\Lambda_N^K(z)\right|\right\}\right. \\
&\quad \left.\cup \left\{\phi; \left|\{x \in \Lambda_N^K(z); \phi_x \leq \left(\frac{1}{2}-\delta\right)L\right\}\right| \geq \frac{1}{2}\eta\left|\Lambda_N^K(z)\right|\right\}\right),
\end{aligned}$$

and $G^K \uparrow G$ as $K \uparrow \infty$, we obtain (1.3).

Proof of Lemma 3.1. We first prove (3.1). Set

$$F_N^{K,+}(r) = \left\{x \in \Lambda_N^K; \phi_x \geq \frac{1}{2}rL\right\},$$

$$\bar{F}_N^{K,+}(r) = \left\{x \in \Lambda_N^L; M_x(\phi) \geq \frac{1}{2}rL\right\}.$$

Then we have

$$\begin{aligned}
&\left\{\left|F_N^{K,+}(r)\right| \geq \eta\left|\Lambda_N^K\right|\right\} \\
&\subset \left\{\left|\bar{F}_N^{K,+}(r')\right| \geq \eta'\left|\Lambda_N^K\right|\right\} \cup \left\{\left|F_N^{K,+}(r) \cap (\bar{F}_N^{K,+}(r'))^c\right| \geq (\eta-\eta')\left|\Lambda_N^K\right|\right\},
\end{aligned}$$

for every $0 < \eta' < \eta < 1$ and $r, r' > 0$. Therefore,

$$
\begin{aligned}
&P_N^{\psi}\left(\left| \left\{ x \in \Lambda_N^K ; \phi_x \geq \frac{1}{2} r L \right\} \right| \geq \eta \left| \Lambda_N^K \right| \big| \mathcal{W}_N(0, L) \right) \\
&\quad \leq P_N^{\psi}\left(\left| \bar{F}_N^{K,+}(r') \right| \geq \eta' \left| \Lambda_N^K \right| \big| \mathcal{W}_N(0, L) \right) \\
&\qquad + P_N^{\psi}\left(\left| F_N^{K,+}(r) \cap (\bar{F}_N^{K,+}(r'))^c \right| \geq (\eta - \eta') \left| \Lambda_N^K \right| \big| \mathcal{W}_N(0, L) \right).
\end{aligned} \tag{3.3}
$$

Now, in a similar manner to the argument of Sec. 3.6 of ref. 10, we can prove that

$$
\begin{aligned}
&P_N\left(\{ \left| \bar{F}_N^{K,+}(r') \right| \geq \eta' \left| \Lambda_N^K \right| \} \cap \mathcal{W}_N(0, L) \right) \\
&\quad \leq P_N\left(\{ \left| \bar{F}_N^{K,+}(r') \right| \geq \eta' \left| \Lambda_N^K \right| \} \cap \Omega_N^-(L) \right) \\
&\quad \leq \exp\left\{ -\frac{C}{L} N^d e^{-\frac{(2-r')^2 L^2}{8 G^K}} \right\},
\end{aligned} \tag{3.4}
$$

for every L, N large enough. Recall that P_N is the measure (1.1) with 0 boundary conditions and coincides with the law of the centered Gaussian field on $\mathbb{R}^{\Lambda_N}$ with covariance matrix $(-\Delta_N)^{-1}$. Combining this estimate with Lemma 2.1 and the lower bound of (1.2), we obtain

$$
\lim_{N \to \infty} P_N^{\psi}\left(\left| \bar{F}_N^{K,+}(r') \right| \geq \eta' \left| \Lambda_N^K \right| \big| \mathcal{W}_N(0, L) \right) = 0,
$$

if $r' > 2 - \frac{\sqrt{G^K}}{\sqrt{G}}$. For the second term of (3.3), we have

$$
\begin{aligned}
&P_N^{\psi}\left(\{ \left| F_N^{K,+}(r) \cap (\bar{F}_N^{K,+}(r'))^c \right| \geq (\eta - \eta') \left| \Lambda_N^K \right| \} \cap \mathcal{W}_N(0, L) \right) \\
&\leq e^{C N^{d-1} L^2} P_N\left(\left| \{ x \in \Lambda_N^K ; \phi_x - M_x(\phi) \geq \frac{1}{2}(r - r') L \} \right| \geq (\eta - \eta') \left| \Lambda_N^K \right| \right),
\end{aligned}
$$

and by LDP estimate we can show that if $r > r'$ then for every L large enough, there exists $N' = N'(L)$ such that this probability is less than e^{-CN^d} for every $N \geq N'$. Therefore the second term in the right hand side of (3.3) is negligible and we obtain (3.1). For (3.2) we consider events

$$
F_N^{K,-}(r) = \left\{ x \in \Lambda_N^K ; \phi_x \leq \frac{1}{2} r L \right\},
$$

$$
\bar{F}_N^{K,-}(r) = \left\{ x \in \Lambda_N^K ; M_x(\phi) \leq \frac{1}{2} r L \right\},
$$

instead of $F_N^{K,+}(r)$, $\bar{F}_N^{K,+}(r)$ and use $\Omega_N^+(0)$ instead of $\Omega_N^-(L)$ in the argument of (3.4). Then, the same argument to the proof of (3.1) yields the result. □

4. TWO INTERFACES BETWEEN TWO WALLS

In this section, we shall prove Theorem 1.2. We first give the lower bound estimate of the probability $P_N^{\psi^1} \otimes P_N^{\psi^2}(\mathcal{W}_N^2(0, L))$.

Proposition 4.1. *For every $\gamma > 0$, there exists $L' > 0$ large enough such that the following holds: for every $L \geq L'$, there exists $N' = N'(L)$ and it holds that*

$$P_N^{\psi^1} \otimes P_N^{\psi^2}\big(\mathcal{W}_N^2(0, L)\big) \geq e^{-N^d e^{-\left(\frac{(\sqrt{2}-1)^2}{4G} - \gamma\right)L^2}},$$

for every $N \geq N'$.

For this proof, we follow an argument similar to that of Proposition 1.1. Let $\theta = \theta(L)$ be a mesoscopic scale which goes to infinity as $L \to \infty$. For the lower interface we first shift its boundary conditions from ψ^1 to 0 and then consider pinning of each site of $\Gamma_N = [\theta]\mathbb{Z}^d \cap \Lambda_N$ to level 0. For the upper interface, we first shift its boundary conditions from ψ^2 to L and then consider pinning of each site of Γ_N to level L. We denote the corresponding measures as Q_N^0 and Q_N^L, respectively. Then, in the same way as the proof of Lemma 2.1 and 2.2, we can obtain the following:

Lemma 4.1. *There exists a constant $C > 0$ independent of L, N such that*

$$P_N^{\psi^1} \otimes P_N^{\psi^2}\big(\mathcal{W}_N^2(0, L)\big) \geq e^{-C(N^{d-1}L^2 + N^d\theta^{-d}L^2)} Q_N^0 \otimes Q_N^L\big(\mathcal{W}_N^2(0, L)\big),$$

for every L, N large enough.

Now, we have that

$$\begin{aligned}
&Q_N^0 \otimes Q_N^L\big(\mathcal{W}_N^2(0, L)\big)\\
&= Q_N^0\big(\Omega_N^{1,+}(0)\big)\, Q_N^L\big(\Omega_N^{2,-}(L)\big)\\
&\quad \times Q_N^0 \otimes Q_N^L\big(\phi_x^1 \leq \phi_x^2 \quad \text{for every} \quad x \in \Lambda_N | \Omega_N^{1,+}(0) \cap \Omega_N^{2,-}(L)\big) \qquad (4.1)\\
&= Q_N^0\big(\Omega_N^{1,+}(0)\big)^2\\
&\quad \times Q_N^0 \otimes Q_N^L\big(\phi_x^1 \leq \phi_x^2 \quad \text{for every} \quad x \in \Lambda_N | \Omega_N^{1,+}(0) \cap \Omega_N^{2,-}(L)\big),
\end{aligned}$$

where

$$\Omega_N^{1,+}(0) = \big\{\phi_x^1 \geq 0 \quad \text{for every} \quad x \in \Lambda_N\big\},$$

$$\Omega_N^{2,-}(L) = \big\{\phi_x^2 \leq L \quad \text{for every} \quad x \in \Lambda_N\big\}.$$

The last equality follows from $Q_N^0\big(\Omega_N^{1,+}(0)\big) = Q_N^L\big(\Omega_N^{2,-}(L)\big)$ which is a consequence of shifting the boundary conditions and turning the variables upside down. We estimate the last term of the right hand side of (4.1).

Lemma 4.2. *Assume that $L > 2\sqrt{2dG\log\theta}$. For every $\gamma > 0$, there exists $L' > 0$ large enough such that the following holds: for every $L \geq L'$, there exists $N' = N'(L)$ and it holds that*

$$Q_N^0 \otimes Q_N^L\big(\phi_x^1 \leq \phi_x^2 \quad \text{for every} \quad x \in \Lambda_N | \Omega_N^{1,+}(0) \cap \Omega_N^{2,-}(L)\big)$$
$$\geq \exp\left\{-N^d \exp\left\{-\frac{(L - 2\sqrt{2dG\log\theta}(1+\gamma))^2}{4G}\right\}\right\},$$

for every $N \geq N'$.

Proof: At first we have

$$\inf_{x\in\Lambda_N} E^{Q_N^0\otimes Q_N^L}\big[\phi_x^2 - \phi_x^1 \big| \Omega_N^{1,+}(0) \cap \Omega_N^{2,-}(L)\big]$$
$$= L - 2\sup_{x\in\Lambda_N} E^{Q_N^0}\big[\phi_x^1 \big| \Omega_N^{1,+}(0)\big]$$
$$\geq L - 2\sqrt{2dG\log\theta}(1+\gamma),$$

for every L, N large enough (N depends on L), where the first equality follows from the change of variables $\phi^2 \leftrightarrow L - \phi^2$ and the last inequality follows from Lemma 2.4. Now, the event $\{\phi_x^1 \leq \phi_x^2$ for every $x \in \Lambda_N\}$ corresponds to the hard wall condition above level 0 for the field $\phi^2 - \phi^1$. Therefore by using FKG inequality and Brascamp-Lieb inequality for the joint measure $Q_N^0(\,\cdot\,|\Omega_N^{1,+}(0)) \otimes Q_N^L(\,\cdot\,|\Omega_N^{2,-}(L))$, the same argument to Lemma 2.5 yields the result. □

Proof of Proposition 4.1. By Lemma 2.3, 4.1, 4.2 and (4.1), taking θ as $\theta = e^{\frac{(\sqrt{2}-1)^2}{4dG}L^2}$ yields the result. □

Proof of Theorem 1.2. By the same reason as the proof of Theorem 1.1, it is sufficient to show the result with Λ_N replaced by Λ_N^K. We first prove the lower bound of (1.4) and (1.5). For the lower bound of (1.4), we have

$$P_N^{\psi^1} \otimes P_N^{\psi^2}\left(\left|\left\{x \in \Lambda_N^K; \phi_x^1 \leq \left(1 - \frac{\sqrt{2}}{2} - \delta\right)L\right\}\right| \geq \eta|\Lambda_N^K| \middle| \mathcal{W}_N^2(0,L)\right)$$
$$\leq e^{CN^{d-1}L^2} P_N^{\psi^1} \otimes P_N^{\psi^2}\Big(\mathcal{W}_N^2(0,L)\Big)^{-1}$$
$$\times P_N\left(\left\{\left|\left\{x \in \Lambda_N^K; \phi_x^1 \leq \left(1 - \frac{\sqrt{2}}{2} - \delta\right)L\right\}\right| \geq \eta|\Lambda_N^K|\right\} \cap \Omega_N^+(0)\right).$$

The argument of the proof of Lemma 3.1 yields that

$$
\begin{aligned}
P_N\Bigg(\Bigg\{\Bigg|\Bigg\{x \in \Lambda_N^K; \phi_x^1 \le \Bigg(1 - \frac{\sqrt{2}}{2} - \delta\Bigg)L\Bigg\}\Bigg| \ \ge \eta\big|\Lambda_N^K\big|\Bigg\} \cap \Omega_N^+(0)\Bigg) \\
\le \exp\Bigg\{-\frac{C}{L}N^d e^{-\frac{1}{2G}(1-\frac{\sqrt{2}}{2}-\frac{\delta}{2})^2L^2}\Bigg\} + \exp\{-CN^d\},
\end{aligned} \tag{4.2}
$$

for L, N large enough. Therefore, by taking γ as $\gamma < \frac{\delta}{2}$ in Proposition 4.1, we obtain the lower bound of (1.4). For the lower bound of (1.5), we have

$$
\begin{aligned}
&P_N^{\psi^1} \otimes P_N^{\psi^2}\Big(\big|\big\{x \in \Lambda_N^K; \phi_x^2 - \phi_x^1 \le (\sqrt{2} - 1 - \delta)L\big\}\big| \ge \eta\big|\Lambda_N^K\big| \ \Big| \ \mathcal{W}_N^2(0, L)\Big) \\
&\le e^{CN^{d-1}L^2} P_N^{\psi^1} \otimes P_N^{\psi^2}\Big(\mathcal{W}_N^2(0, L)\Big)^{-1} \\
&\quad \times P_N \otimes P_N\Big(\big\{\big|\big\{x \in \Lambda_N^K; \phi_x^2 - \phi_x^1 \le (\sqrt{2} - 1 - \delta)L\big\}\big| \ge \eta\big|\Lambda_N^K\big|\big\} \\
&\qquad\qquad \cap\big\{\phi_x^2 - \phi_x^1 \ge 0 \text{ for every } x \in \Lambda_N\big\}\Big).
\end{aligned}
$$

Since the field $\phi^2 - \phi^1$ under $P_N \otimes P_N$ is a centered Gaussian field with covariance matrix $2(-\Delta_N)^{-1}$, by change of variables $\phi \leftrightarrow \frac{1}{\sqrt{2}}\phi$, we can obtain the same estimate as (4.2) for the last term. Therefore by using Proposition 4.1 as before, we obtain

$$
\lim_{N\to\infty} P_N^{\psi^1} \otimes P_N^{\psi^2}\Big(\big|\big\{x \in \Lambda_N^K; \phi_x^2 - \phi_x^1 \le (\sqrt{2} - 1 - \delta)L\big\}\big| \ge \eta|\Lambda_N^K| \ \Big| \ \mathcal{W}_N^2(0, L)\Big) = 0,
$$

for every L large enough and combining this equality with the lower bound of (1.4), we obtain the lower bound of (1.5).

For the upper bound, we have only to shift the boundary conditions, turn the interfaces upside down and repeat the same argument.

APPENDIX A: NOTE ADDED IN PROOF

After the submission of the paper, Yvan Velenik told the author that the lower bound of Proposition 1.1 and Lemma 2.7 can be proved also by the application of Griffith's inequality for a Gaussian lattice field (cf. Appendix A of refs. 9 and 11). However, we would like to stress that our pinning argument works well also for two layered interfaces between two walls. Griffith's inequality cannot be applied in this case.

ACKNOWLEDGMENTS

The author would like to thank Giambattista Giacomin for his helpful advises and Yvan Velenik for informing Griffith's inequality and interesting discussions. He also thanks anonymous referees for their useful remarks and careful reading of the paper. This work was partially supported by JSPS Postdoctoral Fellowships for Research Abroad.

REFERENCES

1. D. Bertacchi and G. Giacomin, Wall repulsion and mutual interface repulsion: a harmonic crystal model in high dimensions, *Stoch. Proc. Appl.* **110**:45–66 (2004).
2. E. Bolthausen and J.–D. Deuschel, Critical large deviation for Gaussian field in the phase transition regime, *Ann. Prob.* **21**:1876–1920 (1993).
3. E. Bolthausen, J.–D. Deuschel, and G. Giacomin, Entropic repulsion and the maximum of two dimensional harmonic crystal, *Ann. Prob.* **29**:1670–1692 (2001).
4. E. Bolthausen, J.–D. Deuschel, and O. Zeitouni, Entropic repulsion of the lattice free field, *Comm. Math. Phys.* **170**:417–443 (1995).
5. E. Bolthausen and D. Ioffe, Harmonic crystal on the wall: a microscopic approach, *Commun. Math. Phys.* **187**:523–566 (1997).
6. J. Bricmont, A. El Mellouki, and J. Fröhlich, Random surfaces in Statistical Mechanics- roughening, rounding, wetting, *J. Stat. Phys.* **42**(5/6):743–798 (1986).
7. J.–D. Deuschel, Entropic repulsion of the lattice free field. II. The 0-boundary case, *Comm. Math. Phys.* **181**: 647–665 (1996).
8. J.-D. Deuschel and G. Giacomin, Entropic repulsion for the free field: pathwise characterization in $d \geq 3$, *Comm. Math. Phys.* **206**:447–462 (1999).
9. F. Dunlop, J. Magnen, V. Rivasseau, and P. Roche, Pinning of an interface by a weak potential, *J. Stat. Phys.* **66**:71–98 (1992).
10. G. Giacomin, Aspects of statistical mechanics of random surfaces, notes of the lectures given at IHP in the fall 2001, Preprint (available at the web page of the author).
11. J. Ginibre, General formulation of Griffith's inequalities, *Comm. Math. Phys.* **16**:310–328 (1970).
12. G. F. Lawler, Intersections of Random Walks, Birkhaüser, 1991.
13. H. Sakagawa, Entropic repulsion for two dimensional multi-layered harmonic crystals, *J. Stat. Phys.* **114**:37–49 (2004).
14. Y. Velenik, Entropic repulsion of an interface in an external field, *Probab. Theory Relat. Fields* **129**:83–112 (2004).
15. Y. Velenik, Localization and delocalization of random interfaces, lecture notes for a minicourse at the meeting "Topics in Random Interfaces and Directed Polymers", Leipzig 2005, arxiv: math. PR/0509695.